Models of Hyperbolic Geometry

Matthew D. Staley
San Francisco State University

Abstract: Hyperbolic geometry naturally arises from the context of Special Relativity. Spacetime is modeled as a 4 dimensional real manifold. Using the Lorenztian inner product on this manifold, we obtain the geometry of a hyperbola. We will investigate the nature of real and complex hyperbolic geometry in a more general setting. We are mostly concerned with the Poincare Disc Model, its tangent space, and whether or not we have a conformal mapping.

1. Real Hyperbolic Models

We begin our study in the ambient space $\mathbb{R}^{n+1}$, with $\mathbf{x} = (x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$. Denote the *Euclidean Norm* for $\mathbf{x} \in \mathbb{R}^{n+1}$ as $||\mathbf{x}||^2 = \sum_{k=1}^{n+1} x_k^2$.

Definition 1. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+1}$. *The Lorentzian inner product is defined to be*

$$< \mathbf{x}, \mathbf{y} >_{(n,1)} = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}.$$

This is a symmetric bilinear form and is allowed to take on negative values, unlike usual inner-products. In fact, Hyperbolic Space is defined for such constant negative values. The $(n, 1)$ denotes the signature.

Definition 2. *Real Hyperbolic-n space is the set*

$$\mathbf{H}^n_{\mathbb{R}} = \{\mathbf{x} \in \mathbb{R}^{n+1} \ : < \mathbf{x}, \mathbf{x} >_{(n,1)} = -1\} \subset \mathbb{R}^{n+1}.$$

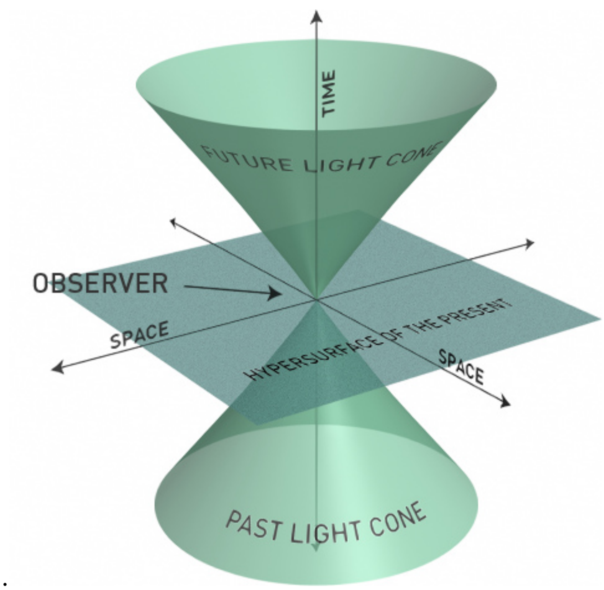

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Figure 1: Minkowski Spacetime

Example 1. *When* $n = 3$ *we have* $\mathbf{H}^3_{\mathbb{R}}$ *as a model of Minkowski Spacetime. The Lorentzian inner product equation becomes*

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = -1.$$

We can think of $(x_1, x_2, x_3) \in \mathbb{R}^3$ *as spacial coordinates and* $x_4 \in \mathbb{R}$ *as time. If we fix one of the spacial coordinates at the "origin", say* $x_1 = 0$ *(such as a fixed observer), then the resulting equation becomes* $x_2^2 + x_3^2 - x_4^2 = -1$*. This is just a hyperboloid in two sheets whose boundary is the "light cone", which is where* $< \mathbf{x}, \mathbf{x} >_{(n,1)} = 0$*. (See Figure 1 above, Source: Wikipedia)*

We now wish to model $\mathbf{H}^n_{\mathbb{R}}$ for any n. We are only interested in the positive, or upper hyperboloid sheet, of the hyperbolic space. There are several ways to do this, such as the upper half plane model and stereographic projection. The following Klein Disc Model is a very intuitive approach to modeling $\mathbf{H}^n_{\mathbb{R}}$ using projections.

1.1. The Klein Disc Model

We start with a vector $\mathbf{x} \in \mathbf{B}^n_{\mathbb{R}} = \{||\mathbf{x}||^2 < 1\}$. Think of $\mathbf{B}^n_{\mathbb{R}}$ embedded into $\mathbb{R}^{n+1}$ at $x_{n+1} = 1$. We want to map $\mathbf{x}$ into $\mathbf{H}^n \subset \mathbb{R}^{n+1}$ by projecting from the origin, through $\mathbf{B}^n$ and into $\mathbf{H}^n_{\mathbb{R}}$. We need to scale the vector appropriately so that $\mathbf{x} \mapsto (\lambda x_1, \ldots, \lambda x_n, \lambda) \in \mathbf{H}^n_{\mathbb{R}}$. To find the λ that works, we simply plug it into the Lorenztian inner product equation and solve for it:

$$\begin{aligned}(\lambda x_1)^2 + \cdots + (\lambda x_n)^2 - \lambda^2 &= -1\\ \lambda^2(||\mathbf{x}||^2 - 1) &= -1\\ \lambda^2 &= \frac{-1}{||\mathbf{x}||^2 - 1}\\ \lambda^2 &= \frac{1}{1 - ||\mathbf{x}||^2}.\end{aligned}$$

We want the upper hyperboloid, so we take the positive square root to obtain

$$\lambda = \frac{1}{\sqrt{1 - ||\mathbf{x}||^2}}.$$

Definition 3. *Let* $\Phi : \mathbf{B}^n_{\mathbb{R}} \to \mathbf{H}^n_{\mathbb{R}}$ *by*

$$\mathbf{x} \mapsto \left(\frac{x_1}{\sqrt{1 - ||\mathbf{x}||^2}}, \frac{x_2}{\sqrt{1 - ||\mathbf{x}||^2}}, \ldots, \frac{x_n}{\sqrt{1 - ||\mathbf{x}||^2}}, \frac{1}{\sqrt{1 - ||\mathbf{x}||^2}} \right).$$

See Figure 2 on next page.

1.1.1. The Columns of $D\Phi$

Note that $\Phi(\mathbf{x})$ is now a vector sitting in $\mathbb{R}^{n+1}$, and each vector component is a function, say f, of n variables. We want to compute $D\Phi = \left[\frac{\partial f_i}{\partial x_j}\right]$ for $i = 1, \ldots, n+1$ and $j = 1, \ldots, n$. Thus $D\Phi$ is an $(n+1) \times (n)$ matrix.

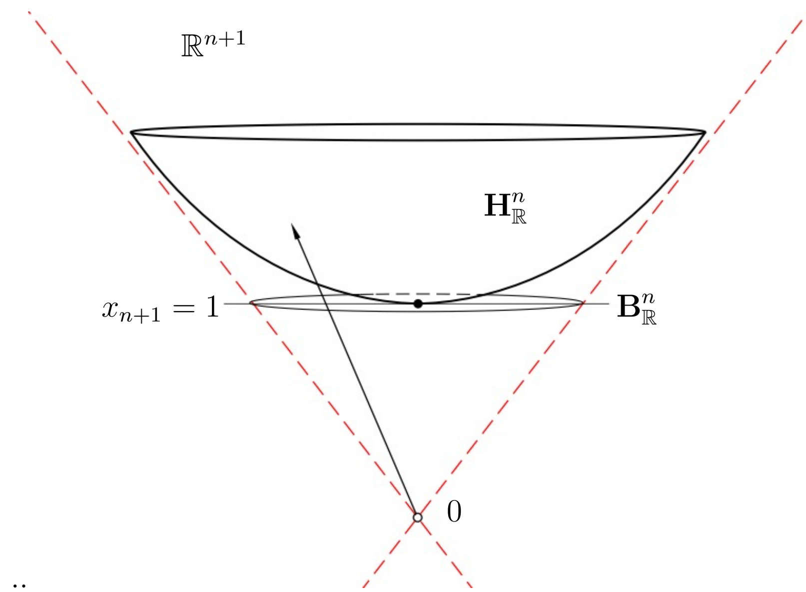

Figure 2: Klein Model

Proposition 1. *Denote* $f_i(x_1, \ldots, x_n) = f_i = \frac{x_i}{\sqrt{1-||\mathbf{x}||^2}}$ *for* $i = 1, \ldots, n$ *and* $f_{n+1} = \frac{1}{\sqrt{1-||\mathbf{x}||^2}}$. *Then we have the following:*

1. *For* $i \neq j$, $\dfrac{\partial f_i}{\partial x_j} = \dfrac{x_i x_j}{(1 - ||\mathbf{x}||^2\ )^{3/2}}$.

2. *For* $i = j$, $\dfrac{\partial f_i}{\partial x_i} = \dfrac{1 + x_i^2 - ||\mathbf{x}||^2}{(1 - ||\mathbf{x}||^2\ )^{3/2}}$.

3. *For* $i = n+1$, *and* $j = 1, \ldots, n$, $\dfrac{\partial f_{n+1}}{\partial x_j} = \dfrac{x_j}{(1 - ||\mathbf{x}||^2\ )^{3/2}}$.

Proof. These derivatives easily follow from the usual quotient and chain rule formulas from calculus as follows:

1. For $i \neq j$,

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{x_i}{\sqrt{1-||\mathbf{x}||^2}}\right) = \frac{0 - x_i\left(\frac{1}{2}(1-||\mathbf{x}||^2)^{-1/2}(-2x_j)\right)}{1-||\mathbf{x}||^2}$$

$$= \frac{-x_i(-x_j)(1-||\mathbf{x}||^2)^{-1/2}}{1-||\mathbf{x}||^2} = \frac{x_i x_j}{(1-||\mathbf{x}||^2)^{3/2}}.$$

2. For $i = j$,

$$\frac{\partial f_i}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\frac{x_i}{\sqrt{1-||\mathbf{x}||^2}}\right) = \frac{\sqrt{1-||\mathbf{x}||^2} - x_i\left(\frac{1}{2}(1-||\mathbf{x}||^2)^{-1/2}(-2x_j)\right)}{1-||\mathbf{x}||^2}$$

$$= \frac{1}{\sqrt{1-||\mathbf{x}||^2}}\left(\frac{1-||\mathbf{x}||^2+x_i^2}{1-||\mathbf{x}||^2}\right) = \frac{1+x_i^2-||\mathbf{x}||^2}{(1-||\mathbf{x}||^2)^{3/2}}.$$

3. For $i = n+1$, and $j = 1, \ldots, n$

$$\frac{\partial f_{n+1}}{\partial x_j} = \frac{\partial}{\partial x_j}(1-||\mathbf{x}||^2)^{-1/2} = -\frac{1}{2}(1-||\mathbf{x}||^2)^{-3/2}(-2x_j) = \frac{x_j}{(1-||\mathbf{x}||^2)^{3/2}}.$$

$\square$

Definition 4. *The Tangent Space at point $p \in \mathbf{H}^n_{\mathbb{R}}$ will be the span of the column vectors of $D\Phi$, that is:*

$$T_p\, \mathbf{H}^n_{\mathbb{R}} = \ span\{\ f_1, \ldots, f_n\ :\ f_i \in D\Phi\}.$$

1.1.2. The Klein Disc Model is Not Conformal

Proposition 2. *Denote the I^{th} column of $D\Phi$ by α and the J^{th} column by β. Then for $I \neq J$, $< \alpha, \beta >_{(n,1)} \neq 0$.*

Proof. Let us write out what α and β look like as column vectors:

$$\left(1-||\mathbf{x}||^2\right)^{3/2}\alpha = \begin{pmatrix} x_1x_I \\ \vdots \\ x_{I-1}x_I \\ 1+x_I^2-||\mathbf{x}||^2 \\ x_{I+1}x_I \\ \vdots \\ x_{J-1}x_I \\ x_Jx_I \\ x_{J+1}x_I \\ \vdots \\ x_nx_I \\ x_I \end{pmatrix}, \qquad \left(1-||\mathbf{x}||^2\right)^{3/2}\beta = \begin{pmatrix} x_1x_J \\ \vdots \\ x_{I-1}x_I \\ x_Ix_J \\ x_{I+1}x_J \\ \vdots \\ x_{J-1}x_J \\ 1+x_J^2-||\mathbf{x}||^2 \\ x_{J+1}x_J \\ \vdots \\ x_nx_J \\ x_J \end{pmatrix}.$$

Then using the Lorentzian inner product on α and β,

$$\left(1-||\mathbf{x}||^2\right)^3 < \alpha,\beta >_{(n,1)}$$

$$= x_Ix_J(x_1^2+\cdots+x_{I-1}^2+1+x_I^2-||\mathbf{x}||^2+x_{I+1}^2+\dots$$
$$+x_{J+1}+1+x_J^2-||\mathbf{x}||^2+x_{J+1}^2+\cdots+x_n^2-1)$$

$$= x_Ix_J\left(||\mathbf{x}||^2+1+1-1-||\mathbf{x}||^2-||\mathbf{x}||^2\right)$$

$$= x_Ix_J\left(1-||\mathbf{x}||^2\right).$$

$$\therefore \quad \boxed{< \alpha,\beta >_{(n,1)} = \frac{x_Ix_J}{(1-||\mathbf{x}||^2)}}$$

Note that this value is dependent on the last coordinate of the column vectors chosen. These two are only zero whenever $\mathbf{x} = (0, \ldots, 0, 0)$. But this is not in the domain of Φ, and so $< \alpha, \beta >_{(n,1)} \neq 0$ for any $\alpha, \beta \in D\Phi$. $\square$

Corollary 3. *The columns of $D\Phi$ are not perpendicular, and so angles are distorted. Thus the Klein Disc model is not conformal.*

1.2. Poincare Disc Model

In this model, we want to map a vector $\mathbf{x} \in \mathbf{B}^n_{\mathbb{R}}$, this time centered at $x_{n+1} = 0$, to $\mathbf{H}^n_{\mathbb{R}}$. To do this, we project from the point $(0, \ldots, 0, 1)$ through $\mathbf{B}^n_{\mathbb{R}}$ to a point $t\mathbf{x} \in \mathbf{H}^n_{\mathbb{R}}$, where $t > 0$. Thus we have that $\mathbf{y} = (tx_1, \ldots, tx_n, t-1) \in \mathbf{H}^n_{\mathbb{R}}$ if and only if

$$\begin{aligned}
(tx_1)^2 + \cdots + (tx_n)^2 - (t-1)^2 &= -1 \\
t^2\left(||\mathbf{x}||^2\right) - t^2 + 2t - 1 &= -1 \\
t^2\left(||\mathbf{x}||^2 - 1\right) &= -2t \\
t &= \frac{2}{1 - ||\mathbf{x}||^2}.
\end{aligned}$$

Definition 5. *Let $\Psi : \mathbf{B}^{n+1}_{\mathbb{R}} \to \mathbf{H}^n_{\mathbb{R}}$ by*

$$\Psi(\mathbf{x}) = \left(\frac{2x_1}{1 - ||\mathbf{x}||^2}, \ \ldots, \frac{2x_n}{1 - ||\mathbf{x}||^2}, \ \frac{1 + ||\mathbf{x}||^2}{1 - ||\mathbf{x}||^2}\right).$$

See Figure 2 on next page.

1.2.1. The columns of $D\Psi$

As before, we have a function of n variables for each vector component in $\Psi(\mathbf{x})$. Then we can compute $D\Psi = \left[\frac{\partial f_i}{\partial x_j}\right]$ for $i = 1, \ldots, n+1$ and $j = 1, \ldots, n$. So $D\Psi$ will be an $(n+1) \times (n)$ matrix. Again, there are three cases for the entries:

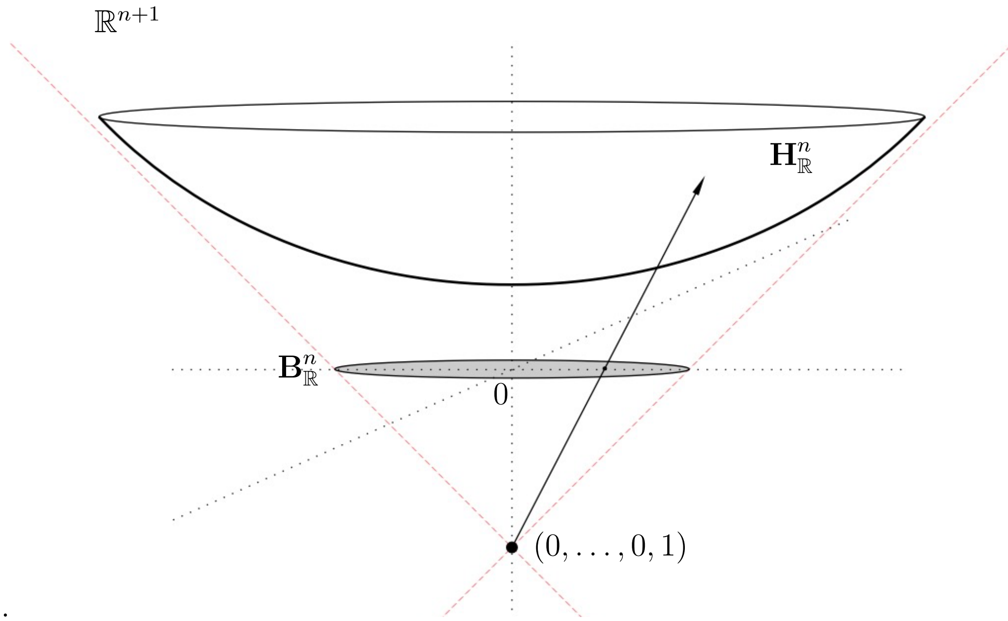

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Figure 3: Poincare Disc Model

Proposition 4. *Denote* $f_i(x_1,\ldots,x_n) = f_i = \frac{2x_i}{1-||\mathbf{x}||^2}$ *for* $i = 1,\ldots,n$ *and* $f_{n+1} = \frac{1+||\mathbf{x}||^2}{(1-||\mathbf{x}||^2)^2}$.

1. *For* $i \neq j$, $\dfrac{\partial f_i}{\partial x_j} = \dfrac{4x_ix_j}{(1-||\mathbf{x}||^2)^2}$.
2. *For* $i = j$, $\dfrac{\partial f_i}{\partial x_i} = \dfrac{2+4x_i^2-2||\mathbf{x}||^2}{(1-||\mathbf{x}||^2)^2}$.
3. *For* $i = n+1$ *and* $j = 1,\ldots,n$, $\dfrac{\partial f_{n+1}}{\partial x_j} = \dfrac{4x_j}{(1-||\mathbf{x}||^2)^2}$.

Proof. 1. For $i \neq j$,

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{2x_i}{1-||\mathbf{x}||^2}\right) = \frac{0-(2x_i)(-2x_j)}{(1-||\mathbf{x}||^2)^2} = \frac{4x_ix_j}{1-||\mathbf{x}||^2}.$$

2. For $i = j$,

$$\frac{\partial}{\partial x_i}\left(\frac{2x_i}{1-||\mathbf{x}||^2}\right) = \frac{(1-||\mathbf{x}||^2)(2)-(2x_i)(-2x_i)}{(1-||\mathbf{x}||^2)^2} = \frac{2+4x_i^2-2||\mathbf{x}||^2}{(1-||\mathbf{x}||^2)^2}.$$

3. For $i = n+1$ and $j = 1, \ldots, n$,

$$\frac{\partial f_{n+1}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{1+||\mathbf{x}||^2}{1-||\mathbf{x}||^2}\right) = \frac{(1-||\mathbf{x}||^2)(2x_j)-(1+||\mathbf{x}||^2)(-2x_j)}{(1-||\mathbf{x}||^2)^2}$$

$$= \frac{2x_j - 2||\mathbf{x}||^2 + 2x_j + ||\mathbf{x}||^2}{(1-||\mathbf{x}||^2)^2} = \frac{4x_j}{(1-||\mathbf{x}||^2)^2}.$$

□

Definition 6. *The Tangent Space at point $p \in \mathbf{H}^n_{\mathbb{R}}$ will be the span of the column vectors of $D\Psi$, that is:*

$$T_p\ \mathbf{H}^n_{\mathbb{R}} = \ span\{\ g_1, \ldots, g_n\ :\ g_i \in D\Psi\}.$$

1.2.2. The Poincare Disc Model is Conformal

Proposition 5. *Denote the I^{th} column of $D\Phi$ by η and the J^{th} column by ξ. Then for $I \neq J$, $<\eta, \xi>_{(n,1)} \equiv 0$.*

Proof. Let us write out what η and ξ look like as column vectors:

$$\left(1-||\mathbf{x}||^2\right)^2\ \eta = \begin{pmatrix} 4x_1x_I \\ \vdots \\ 4x_{I-1}x_I \\ 2+4x_I^2-2||\mathbf{x}||^2 \\ 4x_{I+1}x_I \\ \vdots \\ 4x_{J-1}x_I \\ 4x_Jx_I \\ 4x_{J+1}x_I \\ \vdots \\ 4x_nx_I \\ 4x_I \end{pmatrix}, \qquad \left(1-||\mathbf{x}||^2\right)^2\ \xi = \begin{pmatrix} 4x_1x_J \\ \vdots \\ 4x_{I-1}x_I \\ 4x_Ix_J \\ 4x_{I+1}x_J \\ \vdots \\ 4x_{J-1}x_J \\ 2+4x_J^2-2||\mathbf{x}||^2 \\ 4x_{J+1}x_J \\ \vdots \\ 4x_nx_J \\ 4x_J \end{pmatrix}.$$

Then using the Lorentzian inner product on η and ξ we get:

$$\begin{aligned} &\left(1-||\mathbf{x}||^2\right)^4\ <\eta,\xi>_{(n,1)} \\ &= 16x_Ix_J(x_1^2+\cdots+x_{I-1}^2+\frac{1}{2}+x_I^2-\frac{1}{2}||\mathbf{x}||^2+x_{I+1}^2+\dots \\ &\qquad +\cdots+x_{J-1}^2+\frac{1}{2}+x_J^2-\frac{1}{2}||\mathbf{x}||^2+x_{J+1}^2+\cdots+x_n^2-1) \\ &= 16x_Ix_J\left(||\mathbf{x}||^2-||\mathbf{x}||^2+1-1\right)\equiv 0. \end{aligned}$$

Thus we have $<\eta,\xi>_{(n,1)}\equiv 0$, which further tells us that the columns of $D\Psi$ are orthogonal with respect to the Lorenztian inner product. □

Proposition 6. Ψ *is a conformal mapping.*

Proof. We have already shown that the columns of $D\Psi$ are orthogonal. We want to show that all the columns have the same length. Let $\eta \in D\Psi$ as above. Note that when column I equals row I we have

$$(2+4x_I^2-2||\mathbf{x}||^2)^2 = 16x_I^4 - 16x_I^2||\mathbf{x}||^2 + 16x_I^2 + 4||\mathbf{x}||^4 - 8||\mathbf{x}||^2 + 4.$$

$$\therefore \ \left(1-||\mathbf{x}||^2\right)^4 \ < \eta, \eta >_{(n,1)}$$

$$= 16x_I^2(x_1^2+\cdots+x_{I-1}^2+x_{I+1}^2+\cdots+x_n^2-1) + \\ + \ 16x_I^4 - 16x_I^2||\mathbf{x}||^2 + 16x_I^2 + 4||\mathbf{x}||^4 - 8||\mathbf{x}||^2 + 4$$

$$= 16x_I^2\left(\sum_{k=1}^{n} x_k^2 - ||\mathbf{x}||^2 - 1\right) + 16x_I^2 + 4||\mathbf{x}||^4 - 8||\mathbf{x}||^2 + 4$$

$$= -16x_I^2 + 16x_I^2 + 4(1-2||\mathbf{x}||^2 + ||\mathbf{x}||^4)$$

$$= 4(\ 1-||\mathbf{x}||^2\ )^2.$$

$$\therefore \ \boxed{<\ \eta, \eta >_{(n,1)} \ = \ \frac{4}{(\ 1-||\mathbf{x}||^2\ )^2}}$$

This is a constant length for any choice of $\eta \in D\Psi$. Thus we have that Ψ is a conformal mapping. □

Corollary 7. *The Riemannian Metric on the Poincare Disc Model is*

$$g = \frac{4\ \sum_{k=1}^n\ dx_i^2}{(1-||\mathbf{x}||^2)^2}.$$

2. Projective Spaces and Complex Structures

Before we begin our investigation of hyperbolic $n-$space in the complex case, we layout a few necessary definitions and examples about Projective Spaces.

2.1. Projective Spaces

Definition 7. *Let $\mathcal{V}$ be a vector space. The projective space $\mathbb{P}(\mathcal{V})$ associated to $\mathcal{V}$, is the space of all lines through $\mathcal{V}$. We denote a point in $\mathbb{P}(\mathcal{V})$ corresponding to the line spanned by a nonzero vector $v \in \mathcal{V}$ by $[v]$.*

Example 2 (Riemann Sphere)**.** *The Riemann Sphere is the complex plane plus a point at infinity. We associate this with the complex projective line $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.*

Example 3. *Let $\mathcal{V}$ be a complex vector space of dimension $n+1$. Every line in $\mathbb{C}^{n+1}$ will intersect the unit $2n+1$ sphere in a circle. Thus $\mathbb{P}(\mathcal{V}) = S^{2n+1}/U(1)$. In other words, $\mathbb{P}(\mathcal{V})$ is the quotient space of the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$, under the action of the unitary group $U(1)$. It should pointed out that $U(1)$ is homeomorphic to S^1, since for any $z \in U(1)$, $|z|^2 = 1$.*

2.2. Complex Structure

Let $\mathcal{V}$ be a complex vector space. Then there is a real vector subspace, $\mathcal{V}_{\mathbb{R}} = \{\lambda v \ : \ \lambda \in \mathbb{R}, v \in \mathcal{V}\}$. We can define an automorphism on $\mathcal{V}_{\mathbb{R}}$ by scalar multiplication by $i = \sqrt{-1}$. Denote this map $\mathbb{J} : \mathcal{V}_{\mathbb{R}} \to \mathcal{V}_{\mathbb{R}}$ by $v \mapsto iv$. Note that this map satisfies $\mathbb{J}^2 = -\mathcal{I}$, where $\mathcal{I}$ is the identity map (matrix) and $\mathbb{J}^2 = \mathbb{J} \circ \mathbb{J}$.

Definition 8. *An endomorphism, $\mathbb{J}$, of a real vector space that satisfies $\mathbb{J}^2 = -\mathcal{I}$, is called a linear complex structure.*

Definition 9. *Let $\mathcal{M}$ be a smooth manifold. An almost complex structure $\mathbb{J}$ on $\mathcal{M}$ is a linear complex structure on each tangent space of the manifold,* *which varies smoothly on the manifold. That is* $\mathbb{J} : T\mathcal{M} \to T\mathcal{M}$ *is such that* $\mathbb{J}^2 = -\mathcal{I}$.

Definition 10. *Let* ω *be any function on a complex vector space. Then* ω *is* $\mathbb{J}$*-Linear if* $\omega(\mathbb{J}(\mathbf{z})) = \mathbb{J}(\omega(\mathbf{z}))$.

Example 4. *Let* $\omega : \mathbb{C}^n/\{0\} \to \mathbb{C}^n$ *by* $(z_1, \ldots, z_n) \mapsto \left(\frac{z_1}{||z_1||}, \ldots, \frac{z_n}{||z_n||}\right)$. *Then* ω *is* $\mathbb{J}$*-Linear:*

$$\begin{aligned}
\omega(\mathbb{J}(z_1, \ldots, z_n)) &= \omega(iz_1, \ldots, iz_n) \\
&= \left(\frac{iz_1}{||iz_1||}, \ldots, \frac{iz_n}{||iz_n||}\right) \\
&= \left(\frac{iz_1}{|i| \cdot ||z_1||}, \ldots, \frac{iz_n}{|i| \cdot ||z_n||}\right) \\
&= \left(i\frac{z_1}{||z_1||}, \ldots, i\frac{z_n}{||z_n}\right) \\
&= \mathbb{J}\left(\frac{z_1}{||z_1||}, \ldots, \frac{z_n}{||z_n||}\right) \\
&= \mathbb{J}(\omega(z_1, \ldots, z_n)).
\end{aligned}$$

3. Complex Hyperbolic Geometry

We think of $\mathbb{C}^{n+1}$ as $\mathbb{C}^n \times \mathbb{C}$. For $\mathbf{z} \in \mathbb{C}^{n+1}$, we write $\mathbf{z} = (z_1, \ldots, z_n, z_{n+1})$, where each $z_k = x_k + iy_k \in \mathbb{C}$, for some $x_k, y_k \in \mathbb{R}$. In the real hyperbolic case we used the Lorentzian inner product, which was a symmetric bilinear form. To generalize this to the complex case we have the following.

Definition 11. *A Hermitian form on a complex vector space $\mathcal{V}$ is a paring $< v, w > \in \mathbb{C}$ for all $v, w \in \mathcal{V}$, which is bilinear over the reals and satisfies*

1. $\overline{< v, w >} = < w, v >$ *for all* $v, w \in \mathcal{V}$ *and*
2. $< \alpha v, w > = \alpha < v, w >$ *for all* $v, w \in \mathcal{V}$, $\alpha \in \mathbb{C}$.

Note that we lose bi-linearity over $\mathbb{C}$ here because for $\alpha \in \mathbb{C}$

$$< v, \alpha w > = \overline{< \alpha w, v >} = \overline{\alpha < w, v >} = \overline{\alpha}\, \overline{< w, v >} = \overline{\alpha} < v, w > .$$

Another consequence is that $< v, v > \in \mathbb{R}\ \forall\ v \in \mathcal{V}$. There are many types of Hermitian forms. We are interested in the form that is analogous to the Lorentzian inner product.

Definition 12. *The First Hermitian Form on $\mathbb{C}^{n+1}$ is*

$$< \mathbf{z}, \mathbf{w} >_{(n,1)} = -z_{n+1}\overline{w_{n+1}} + \sum_{k=1}^{n} z_k \overline{w_k}.$$

Definition 13. *Complex Hyperbolic n-space, denoted $\mathbf{H}^n_{\mathbb{C}}$, is defined to be the subset $\mathbb{P}(\mathbb{C}^{n+1})$ consisting of negative lines in $\mathbb{C}^{n+1}$. As a set, we have*

$$\mathbf{H}^n_{\mathbb{C}} = \{[v] \in \mathbb{P}(\mathbb{C}^{n+1}) \ : \ < v, v >_{(n,1)} < 0\}.$$

Definition 14 (Epstein)**.** *The tangent space to a nonzero $[x] \in \mathbf{H}^n_{\mathbb{C}}$ can be identified with*

$$T_{[x]}\ \mathbf{H}^n_{\mathbb{C}} = \{[u] \in \mathbb{P}(\mathbb{C}^{n+1}) \ : < u, x >_{(n,1)} = 0 \ \ \textit{and} \ < u, \mathbb{J}(x) >_{(n,1)} = 0\}.$$

Is there a way to model $\mathbf{H}^n_{\mathbb{C}}$ and its tangent space in local coordinates along with the conformal properties that we desire? The following is an investigation using a similar approach to the Poincare Disc Model.

Definition 15. *Let $\mathcal{M}$ be the manifold whose First Hermitian Form is constantly -1, i.e.*

$$\mathcal{M} = \{\mathbf{w} \in \mathbb{C}^{n+1} \ : < \mathbf{w}, \mathbf{w} >_{(n,1)} \ = \ -1\}.$$

Its tangent space will be

$$T_{\mathbf{w}}\mathcal{M} = \{\mathbf{z} \in \mathbb{C}^{n+1} \ : < \mathbf{z}, \mathbf{w} >_{(n,1)} \ = \ 0\}.$$

If we let $\mathbf{w} = (w_1, \ldots, w_n, w_{n+1}) \in \mathbb{C}^{n+1}$, then then the "hyperbolic equation" for $\mathcal{M}$ is:

$$< \mathbf{w}, \mathbf{w} >_{(n,1)} = -w_{n+1}\overline{w_{n+1}} + \sum_{k=1}^{n} w_k\overline{w_k} = -|w_{n+1}|^2 + \sum_{k=1}^{n} |w_k|^2 = -1.$$

Observe that the manifold $\mathcal{M}$ has real dimension $2n+1$. $\mathcal{M}$ is not a model of $\mathbf{H}^n_{\mathbb{C}}$, but rather contains a model of $\mathbf{H}^n_{\mathbb{C}}$ as a real codimension one submanifold.

On the other hand, $T_{\mathbf{w}}\mathcal{M}$ is a complex vector space of dimension n. The condition that $< \mathbf{z}, \mathbf{w} >_{(n,1)} = 0$ gives us two real equations to solve for: the real and imaginary parts. There are $2n+2$ variables and 2 real conditions, hence $T_{\mathbf{w}}\mathcal{M}$ has real dimension $2n$.

Furthermore, $T_{\mathbf{w}}\mathcal{M}$ does indeed represent the tangent space to $\mathbf{H}^n_{\mathbb{C}}$ (c.c. Epstein), and it is contained in the tangent space to $\mathcal{M}$ at $\mathbf{w}$, which is one dimension higher. Under the real setting, we can use the Lorenztian inner product to study the geometry of $\mathcal{M}$, which will now have signature $(2n, 2)$.

3.1. Complex Poincare Model for $\mathcal{M}$

Definition 16. *Let* $\mathbf{B}^n_{\mathbb{C}} = \{(z_1, \ldots, z_n, 0) \in \mathbb{C}^{n+1} \ : \ |\mathbf{z}|^2 < 1\}$, *where*

$$|\mathbf{z}|^2 = \sum_{k=1}^{n} |z_k|^2 = \sum_{k=1}^{n} z_k\overline{z_k} = \sum_{k=1}^{n} (x_k^2 + y_k^2).$$

Definition 17. *Let* $\Psi : \ \mathbf{B}^n_{\mathbb{C}} \to \mathcal{M}$ *by*

$$\Psi(\mathbf{z}) = \left(\frac{2z_1}{1-|\mathbf{z}|^2}, \dots, \frac{2z_n}{1-|\mathbf{z}|^2}, \frac{1+|\mathbf{z}|^2}{1-|\mathbf{z}|^2} \right).$$

We have that $\Psi(\mathbf{z}) \in \mathcal{M}$ because

$$|w_1|^2 + \cdots + |w_n|^2 - |w_{n+1}|^2 = \frac{4|z_1|^2}{(1-|\mathbf{z}|^2)^2} + \cdots + \frac{4|z_n|^2}{(1-|\mathbf{z}|^2)^2} - \frac{(1+|\mathbf{z}|^2)^2}{(1-|\mathbf{z}|^2)^2}$$

$$= \frac{4|\mathbf{z}|^2 - 1 - 2|\mathbf{z}|^2 - |\mathbf{z}|^4}{(1-|\mathbf{z}|^2)^2} = \frac{-1 + 2|\mathbf{z}|^2 - |\mathbf{z}|^4}{(1-|\mathbf{z}|^2)^2} = \frac{-(|\mathbf{z}|^2-1)^2}{(1-|\mathbf{z}|^2)^2} = -1.$$

3.1.1. Local Coordinates of $\mathcal{M}$

Rewrite each w_k into $u_k + iv_k$, so that for $k = 1, \dots, n$,

$$w_k = \frac{2z_k}{1-|\mathbf{z}|^2} = \frac{2x_k}{1-|\mathbf{z}|^2} + i\frac{2y_k}{1-|\mathbf{z}|^2}.$$

$$\boxed{u_k = \frac{2x_k}{1-|\mathbf{z}|^2}} \qquad \boxed{v_k = \frac{2y_k}{1-|\mathbf{z}|^2}}$$

For $k = n+1$, we have that $w_{n+1} = \dfrac{1+|\mathbf{z}|^2}{1-|\mathbf{z}|^2} \in \mathbb{R}$. Thus

$$\boxed{u_{n+1} = \frac{1+|\mathbf{z}|^2}{1-|\mathbf{z}|^2}} \qquad \boxed{v_{n+1} \equiv 0}$$

Note that the u_k and v_k play a similar role in the real f_i's from the first section. That is, each u_k and v_k are (real) functions in n variables. We are now in position to compute $D\Psi$ in terms of the real and imaginary parts. $D\Psi$ will be a $(2n+2) \times (2n)$ matrix and the terms will be ordered in the following way:

$$\begin{pmatrix}
\partial u_1/\partial x_1 & \partial u_1/\partial y_1 & \dots & \partial u_1/\partial x_n & \partial u_n/\partial y_n \\
\partial v_1/\partial x_1 & \partial v_1/\partial y_1 & \dots & \partial v_1/\partial x_n & \partial v_n/\partial y_n \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\partial u_n/\partial x_1 & \partial u_n/\partial y_1 & \dots & \partial u_n/\partial x_n & \partial u_n/\partial y_n \\
\partial v_n/\partial x_1 & \partial v_n/\partial y_1 & \dots & \partial v_n/\partial x_n & \partial v_n/\partial y_n \\
\partial u_{n+1}/\partial x_1 & \partial u_{n+1}/\partial y_1 & \dots & \partial u_{n+1}/\partial x_n & \partial u_{n+1}/\partial y_n \\
\partial v_{n+1}/\partial x_1 & \partial v_{n+1}/\partial y_1 & \dots & \partial v_{n+1}/\partial x_n & \partial v_{n+1}/\partial y_n
\end{pmatrix}$$

The columns of $D\Psi$

3.1.2. The Columns of $D\Psi$

Proposition 8. *Given $D\Psi$ as above, we have the following column entries:*

1. *The partial derivatives u_k with respect to x_k and y_k are as follows:*

   (a) *For $k = 1, \dots, n$, and $j = 1, \dots, n$, and $j \neq k$, we have:*

$$\frac{\partial u_k}{\partial x_j} = \frac{4x_k x_j}{(1 - |\mathbf{z}|^2)^2}.$$

   (b) *For $j = k$ we have:*

$$\frac{\partial u_k}{\partial x_k} = \frac{2 - 2|\mathbf{z}|^2 + 4x_k^2}{(1 - |\mathbf{z}|^2)^2}.$$

   (c) *For $k = 1, \dots, n$, and $j = 1, \dots, n$, ($j = k$ is allowed), we have*

$$\frac{\partial u_k}{\partial y_j} = \frac{4x_k y_j}{(1 - |\mathbf{z}|^2)^2}.$$

2. *The partial derivatives v_k with respect to x_k and y_k are as follows:*

(a) *For* $k = 1, \ldots, n$, *and* $j = 1, \ldots, n$, *and* $j \neq k$, *we have:*

$$\frac{\partial v_k}{\partial y_j} = \frac{4 y_k y_j}{(1 - |\mathbf{z}|^2)^2}.$$

(b) *For* $j = k$ *we have:*

$$\frac{\partial v_k}{\partial y_k} = \frac{2 - 2|\mathbf{z}|^2 + 4y_k^2}{(1 - |\mathbf{z}|^2)^2}.$$

(c) *For* $k = 1, \ldots, n$, *and* $j = 1, \ldots, n$, *(*$j = k$ *is allowed), we have:*

$$\frac{\partial v_k}{\partial x_j} = \frac{4 y_k x_j}{(1 - |\mathbf{z}|^2)^2}.$$

3. *Lastly, the partial derivatives* v_{n+1} *and* u_{n+1} *with respect to* x_j *and* y_j *for* $j = 1, \ldots, n$ *are as follows:*
   (a) $\dfrac{\partial v_{n+1}}{\partial x_j} \equiv 0.$

   (b) $\dfrac{\partial v_{n+1}}{\partial y_j} \equiv 0.$

   (c) $\dfrac{\partial u_{n+1}}{\partial x_j} = \dfrac{4x_j}{(1 - |\mathbf{z}|^2)^2}.$

   (d) $\dfrac{\partial u_{n+1}}{\partial y_j} = \dfrac{4y_j}{(1 - |\mathbf{z}|^2)^2}.$

Proof. These derivatives are straightforward to check and very similar to the proof worked out in 1.2.1., Prop. 4. □

3.2. The Geometry of $D\Psi$

We want to show that the columns of $D\Psi$ are mutually orthogonal and independent in length, just as in the real case. Then want to show that the First Hermitian Form on all the combinations is zero, and is constant with respect to itself. Equivalently, because the entries of $D\Psi$ are entirely real, we use the Lorentzian inner product with signature $(2n, 2)$, which will be denoted $< \cdot\, ,\ \cdot >_{(2n,2)}$.

To do this, choose an arbitrary I^{th} column vector with respect to ∂u, and denote it μ_I. Similarly choose an arbitrary J^{th} column vector with respect to ∂v, and denote it ν_J. Then write out what the columns look like for $I \neq J$:

$$(1-|\mathbf{z}|^2)^4\ \mu_I = \begin{pmatrix} 4x_1x_I \\ 4y_1x_I \\ \vdots \\ 4x_Jx_I \\ 4y_Jx_I \\ \vdots \\ 2-2|\mathbf{z}|^2+4x_I^2 \\ 4y_Ix_I \\ \vdots \\ 4x_nx_I \\ 4y_nx_I \\ 4x_I \\ 0 \end{pmatrix}, \qquad (1-|\mathbf{z}|^2)^4\ \nu_J = \begin{pmatrix} 4x_1x_J \\ 4y_1x_J \\ \vdots \\ 2-2|\mathbf{z}|^2+4x_J^2 \\ 4y_Jx_J \\ \vdots \\ 4x_Ix_J \\ 4y_Ix_J \\ \vdots \\ 4x_nx_J \\ 4y_nx_J \\ 4x_J \\ 0 \end{pmatrix}.$$

Proposition 9. *Given $D\Psi$ above, we have the following for $I \neq J$:*

1. $< \mu_I, \mu_J >_{(2n,2)}\ =\ 0.$
2. $< \mu_I, \nu_J >_{(2n,2)}\ =\ 0.$
3. $< \nu_I, \nu_J >_{(2n,2)}\ =\ 0.$
4. *The columns of $D\Psi$ are of the same length.*

Proof. Let $I \neq J$, and $I, J = 1, \ldots, n$, then

1. When we multiply the I^{th} and J^{th} columns at row $I = J$ we get

$$4x_I x_J(2 - 2|\mathbf{z}|^2 + 4x_I^2) = 8x_I x_J - 8x_I x_J|\mathbf{z}|^2 + 16x_I^2 x_J^2.$$

$$\therefore\ (1 - |\mathbf{z}|^2)^4\ < \mu_I, \mu_J >_{(2n,2)}$$

$$= 16x_I x_J(x_1^2 + y_1^2 + \cdots + \frac{1}{2} - \frac{1}{2}|\mathbf{z}|^2 + x_J^2 + y_J^2 + \cdots + \frac{1}{2} - \frac{1}{2}|\mathbf{z}|^2 + x_I^2 + y_I^2 + \cdots + x_n^2 + y_n^2 - 1 - 0)$$

$$= 16x_I x_J \left(\left(\sum_{k=1}^{n} x_k^2 + y_k^2 \right) + 1 - |\mathbf{z}|^2 - 1 \right)$$

$$= 16x_I x_j(|\mathbf{z}|^2 - |\mathbf{z}|^2) = 0.$$

$$\therefore\ < \mu_I, \mu_J >_{(2n,2)}\ =\ 0.$$

2. Similarly, at row $I = J$ we get

$$4x_I y_J(2 - 2|\mathbf{z}|^2 + 4y_J^2) = 8x_I y_J - 8|\mathbf{z}|^2 x_I y_J + 16y_J^2 x_I y_J.$$

$$\therefore\ (1 - |\mathbf{z}|^2)^4\ < \mu_I, \nu_J >_{(2n,2)}$$

$$= 16x_I y_J(x_1^2 + y_1^2 + x_J^2 + \frac{1}{2} - \frac{1}{2}|\mathbf{z}|^2 + y_J^2 + \cdots + \frac{1}{2} - \frac{1}{2}|\mathbf{z}|^2 + x_I^2 + y_I^2 + \cdots + x_n^2 + y_n^2 - 1 - 0)$$

$$= 16x_I y_J \left(\left(\sum_{k=1}^{n} x_k^2 + y_k^2 \right) + 1 - |\mathbf{z}|^2 - 1 \right)$$

$$= 16x_I y_J \left(|\mathbf{z}|^2 - |\mathbf{z}|^2 \right) = 0.$$

$\therefore \ < \mu_I, \nu_J >_{2n,n} = \ 0.$

3. If we look at μ_I versus ν_J, they only differ by the last indices. So the computation is exactly the same as before, except we permute $I \to J$ and conclude that $< \nu_I, \nu_J >_{(2n,2)} = \ 0.$

4. We want to calculate $(a) \ < \mu_I, \mu_I >_{(2n,2)}$ and $(b) \ < \nu_J, \nu_J >_{(2n,2)}$ and show that they are independent of the index for $I, J = 1, \ldots, n$.

   (a) First note that when row $I = I$ we get

$$(2 - 2|\mathbf{z}|^2 + 4x_I^2)^2 = 4|\mathbf{z}|^4 - 8|\mathbf{z}|^2 - 16x_I^2|\mathbf{z}|^2 + 16x_I^4 + 16x_I^2 + 4.$$

$$\therefore \ (1 - |\mathbf{z}|^4) \ < \mu_I, \mu_I >_{(2n,2)}$$

$$= \ 16x_I^2(x_1^2 + y_1^2 + \cdots + (2 - 2|\mathbf{z}|^2 + 4x_I^2)^2 + y_I^2 + \cdots + x_n^2 + y_n^2 - 1 - 0)$$

$$= 16x_I^2 \left(\sum_{k=1}^{n} x_k^2 + y_k^2 \right) - 16x_I^2|\mathbf{z}|^2 + 4|\mathbf{z}|^4 - 8|\mathbf{z}|^2 + 4$$

$$= 16x_I^2(|\mathbf{z}|^2 - |\mathbf{z}|^2) + 4(|\mathbf{z}|^4 - 2|\mathbf{z}|^2 + 1)$$

$$= 4(1 - |\mathbf{z}|^2)^2.$$

Thus $< \mu_I, \mu_I >_{(2n,2)} = \ \dfrac{4}{(1 - |\mathbf{z}|^2)^2}$, which is independent.

(b) By a very similar calculation $< \nu_J, \nu_J >_{(2n,2)} = \dfrac{4}{(1-|\mathbf{z}|^2)^2}$.

□

Definition 18. *For any $\alpha \in D\Psi$, we can define a norm on $\mathcal{M}$ as*

$$||\alpha||_{\mathcal{M}} \;=\; \sqrt{< \alpha, \alpha >_{(2n,2)}} \;=\; \sqrt{\frac{4}{(1-|\mathbf{z}|^2)^2}} \;=\; \frac{2}{1-|\mathbf{z}|^2}.$$

3.2.1. Final Remarks

What we have shown is that the pull-back of the $(2n, 2)$-signature form under Ψ is indeed a conformal Riemannian metric. However, the image $\mathcal{N} := \Psi(\mathbf{B}^n_{\mathbb{C}})$ is a codimension one submanifold of $\mathcal{M}$, whose tangent space does not coincide with that of $T_{\mathbf{w}}\mathcal{M}$. Note that Ψ is not $\mathbb{J}-$linear:

$$\Psi(\mathbb{J}(\mathbf{z})) = \left(\frac{2iz_1}{1-|\mathbf{z}|^2}, \cdots, \frac{2iz_n}{1-|\mathbf{z}|^2}, \frac{1+|\mathbf{z}|^2}{1-|\mathbf{z}|^2}\right), \text{ and}$$

$$\mathbb{J}(\Psi(\mathbf{z})) = \left(\frac{2iz_1}{1-|\mathbf{z}|^2}, \cdots, \frac{2iz_n}{1-|\mathbf{z}|^2}, i \cdot \frac{1+|\mathbf{z}|^2}{1-|\mathbf{z}|^2}\right).$$

To get the complex hyperbolic metric, one would need to pull-back the $(2n, 2)$-signature form under the map $\tilde{\Psi} : T_{\mathbf{w}}\mathcal{N} \to T_{\mathbf{w}}\mathcal{M}$, where $\tilde{\Psi} = F \circ \Psi$, and where $F : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is linear. There are many such maps, but it is unclear if there is a nice one that makes it easy to do the calculations.

References

[1] Epstein, Warwick and Durham, Analytical and Geometric Aspects of Hyperbolic Space, London Mathematical Society Lecture Note Series #111, Cambridge University Press, 1987.

[2] Goldman, W.M., Complex Hyperbolic Geometry, Clarendon Press, Oxford, 1999.

[3] Naber, G.L., The Geometry of Minkowski Spacetime, Applied Mathematical Sciences, Volume 92, Springer-Verlag New York, 1992.

[4] Parkkonen, J., Paulin, F., Prescribing the behaviour of geodesics in negative curvature, J. Geometry & Topology 14 (2010) 277-392.

This paper is part of the Master of Arts in Mathematics requirement at San Francisco State University, Department of Mathematics:
TH 937, 1600 Holloway Avenue, SF, CA 94132.
Email: staley.d.matthew@gmail.com